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THE MODULE OF ZARISKI-DIFFERENTIALS OF A NORMAL GRADED GORENSTEIN-SINGULARITY

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1. Introduction

In the papers [5,6] we have studied the properties of pure subrings A or invariant subrings A of a local regular k-algebra B and have tried to find as much as possible total differentials of the finite differential module $D_k(A)$ of A which form part of a minimal system of generators of $(D_k(A))^{**}$ – the so-called module of Zariski-differentials of A. (For the theory of finite differential modules we refer the reader to the work [7] of G. Scheja and U. Storch.) For example, if the field k has characteristic zero and the pure extension $A \to B$, B being regular, is non-degenerate in a suitable sense, then one can find $s = \dim A$ (total) differentials in $D_k(A)$ which form part of a minimal system of generators of $(D_k(A))^{**}$. In case there is a finite group G acting on B with $B^G = A$, then any minimal system of generators of $D_k(A)$ is part of a minimal system of generators of $(D_k(A))^{**}$. In this paper we will prove that this last embedding-property holds for arbitrary *normal graded Gorenstein-singularities over a field of characteristic zero.* We use the term 'graded' in a sense, which we will discuss in the following section.

Let k be a field of characteristic zero; a local analytic k-algebra A with $k = k_A := A/\mathfrak{m}_A$ is called *(positively) graded*, if there exists a k-derivation δ of A and a system of generators $x_1, \dots, x_m$ of the maximal ideal $\mathfrak{m}_A$ of A with $\delta x_i = m_i x_i$ and strict positive eigenvalues $m_i \in \mathbb{N}$. We call δ a *grading derivation of A*. The theory of graded analytic k-algebras has been developed in the works [8–11] of G. Scheja and H. Wiebe. In general, it is very difficult to decide whether a local analytic k-algebra is graded or not. G. Scheja and H. Wiebe have given many such criteria, for example:

1.1. *It suffices to require the identity* $\delta x_i \equiv m_i x_i$ *modulo* $\mathfrak{m}_A^2$ *with* $m_i \in \mathbb{N}$, $m_i > 0$. (See Korollar (2.7) in [10].)

1.2. *If A is a reduced complete intersection with isolated singularity, then it is*

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enough to require that there exists a derivation which acts bijectively on $\mathfrak{m}_A/\mathfrak{m}_A^2$. (See Satz (4.4) in [9].)

1.3. *If k is algebraically closed and A is a normal domain of dimension 2, then it suffices to require that there exists a derivation of A which acts not nilpotently on* $\mathfrak{m}_A/\mathfrak{m}_A^2$. (See Satz (3.1) in [10].)

A non-unit $g \neq 0$ of a local graded k-algebra A with grading derivation δ and char $k=0$ is called *homogeneous of degree* α, if g is an eigenvector of δ with eigenvalue α. Then α is necessarily a positive integer. From [4, p. 144] we obtain immediately:

1.4. *If* $t=$ depth A, *then there exists a homogeneous A-sequence of length t whose elements are all of the same degree.*

2. An embedding-property of the module of Zariski-differentials of a normal graded Gorenstein-singularity

Let us begin with the following lemma:

2.1. Lemma. *Let k be a field of characteristic zero and A be a normal, local graded analytic k-algebra with grading derivation* δ. *Then one of the following assertions is true:*

(a) *The canonical induced map* $\mathrm{D}_k(A)\otimes_A k_A \to (\mathrm{D}_k(A))^{**}\otimes_A k_A$ *is injective.*

(b) *The derivation* δ *generates a free direct summand of the derivation module* $(\mathrm{D}_k(A))^*$ *of* A.

Proof. We may assume dim $A \geq 2$, and we will only use the fact that A has depth ≥ 2. There exists an A-linear surjective map $\mathrm{D}_k(A)\to \mathfrak{m}_A$ with $\mathrm{d}x_i \mapsto m_i x_i$, which we will again call δ. For the corresponding bidual map $\delta^{**}:(\mathrm{D}_k(A))^{**}\to(\mathfrak{m}_A)^{**}=A$ we discuss the following cases:

(a) δ^{**} is not surjective; then it follows from the inclusions

$$\mathfrak{m}_A = \operatorname{im}\delta \subseteq \operatorname{im}\delta^{**} \subsetneq \mathfrak{m}_A$$

that δ^{**} maps $(\mathrm{D}_k(A))^{**}$ *onto* $\mathfrak{m}_A$. Since the composite map $\mathrm{D}_k(A)\to(\mathrm{D}_k(A))^{**}\to\mathfrak{m}_A$ is surjective and therefore bijective modulo $\mathfrak{m}_A$, it follows that the first assertion must hold.

(b) $\delta^{**}:(\mathrm{D}_k(A))^{**}\to A$ is surjective; in this case $(\mathfrak{m}_A)^*=A^*$ is a direct summand of $(\mathrm{D}_k(A))^*$ via the dualized map $\delta^*:(\mathfrak{m}_A)^*=A^*\to(\mathrm{D}_k(A))^*$. Now, one easily sees that $\delta^*(1)=\delta$, and in this case the second assertion holds.

2.2. Remark. In addition to the Lemma 2.1 we observe: *The grading derivation* δ

generates a free direct summand of $(D_k(A))^*$ *if and only if the bidualized map* $\delta^{**}:(D_k(A))^{**}\to(\mathfrak{m}_A)^{**}=A$ *is surjective.* The if-part of this statement has been already proved; the only-if-part is seen as follows: Let $\delta^*:(\mathfrak{m}_A)^*\to(D_k(A))^*$ be the dualized map of $\delta:D_k(A)\to\mathfrak{m}_A$ with $\delta^*(1)=\delta$. If $\delta=\delta^*(1)$ generates a free direct summand of $(D_k(A))^*$, then $\delta^{**}:(D_k(A))^{**}\to(\mathfrak{m}_A)^{**}=A$ must be surjective.

In the two-dimensional case we have:

2.3. Corollary. *Let A be as in the preceding lemma with* $\dim A\le 2$ *and let* ω_A *be the canonical module of A; then one of the following assertions is true:*

(a) *The canonical induced map* $D_k(A)\otimes_A k_A\to(D_k(A))^{**}\otimes_A k_A$ *is injective.*

(b) *The derivation module* $(D_k(A))^*$ *of A is canonically isomorphic to* $A\cdot\delta\oplus\omega_A^*$.

Proof. Let $\dim A=2$; in case (b) of Lemma 2.1 we consider the surjective map $\delta^{**}:(D_k(A))^{**}\to A$ with $U:=\operatorname{Ker}\delta^{**}$, see Remark 2.2. Then U is reflexive of rank 1. It suffices to show that U is the canonical module of A. Since the sequence

$$0\to U\to(D_k(A))^{**}\to A\to 0$$

is split-exact, one easily sees that

$$(\Lambda^2(D_k(A))^{**})^{**}=(\Lambda^1 U)^{**}=U^{**}=U,$$

and $(\Lambda^2(D_k(A))^{**})^{**}=(\Lambda^2 D_k(A))^{**}$ is the canonical module ω_A of A.

Now, if in the situation of the preceding Corollary A is *a normal local graded algebra which is a Gorenstein ring of dimension* 2 *over a field of characteristic zero, then assertion* (b) *cannot hold* (*and therefore assertion* (a) *must hold*), since in the Gorenstein case the canonical module ω_A of A, and hence ω_A^*, is free, and from the canonical decomposition $(D_k(A))^*=A\cdot\delta\oplus\omega_A^*$ we obtain that the derivation module $(D_k(A))^*$ of A is free what (in the graded case) implies that A is regular, see [3,4]. But in the regular case $A=k\langle X_1,\dots,X_s\rangle$ we know that $\delta=\sum_{i=1}^s m_iX_i\cdot\partial_i$, and then δ is *not* part of a free basis of $(D_k(A))^*$ which contradicts the canonical decomposition $(D_k(A))^*=A\cdot\delta\oplus\omega_A^*=A\cdot\delta\oplus A$ of condition (b) in the Gorenstein case.

The following proposition is the main result of this section.

2.4. Proposition. *Let k be a field of characteristic zero and A be a local graded analytic k-algebra which is a normal Gorenstein-singularity.*

Then the canonical induced map $D_k(A)\otimes_A k_A\to(D_k(A))^{**}\otimes_A k_A$ *is injective, i.e. any minimal system of generators of* $D_k(A)$ *forms part of a minimal system of generators of* $(D_k(A))^{**}$.

Proof. We will show that condition (b) of Lemma 2.1 cannot hold. Assume the contrary and choose an example A of minimal dimension $s=\dim A$. We will obtain a contradiction by constructing an example of lower dimension. According to the remarks following Corollary 2.3 we have necessarily $s=\dim A\geq 3$. Let $\delta:\mathrm{D}_k(A)\to \mathfrak{m}_A$ be as in the proof of Lemma 2.1 and let δ be a free direct summand of $(\mathrm{D}_k(A))^*$ by assumption. Then it follows from Remark 2.2 that $\delta^{**}:(\mathrm{D}_k(A))^{**}\to A$ is surjective. Let $\mathfrak{a}\subsetneq A$ be an ideal defining the variety of the singular locus of A. Since A is normal, $\mathfrak{a}$ has codimension ≥ 2. Let $\mathfrak{Q}$ be the finite (and possibly empty) set of prime ideals in A of codimension 2 which contain $\mathfrak{a}$. If $f_1,\dots,f_s$ is a homogeneous A-sequence with $\delta f_i=\alpha f_i$ and $\alpha\in\mathbb{N}$, $\alpha>0$, see 1.4, then by the result (4.6) of H. Flenner in [1] concerning Bertini-theorems there exists a linear combination

$$f=\sum_i c_i f_i,\quad c_i\in k,$$

with the following properties:

(1) $f\notin \mathfrak{q}^{(2)}$ for all non-maximal prime ideals $\mathfrak{q}\subsetneq \mathfrak{m}_A$.

(2) $f\notin\bigcup\mathfrak{Q}$.

It follows that $\delta f=\sum_i c_i\delta f_i=\sum_i c_i\alpha f_i=\alpha f$, and it is easily seen that the ring A/fA is normal (since conditions (1) and (2) hold) and satisfies all the hypotheses of our theorem. Now consider the composite map

$$\mathrm{D}_k(A)/f\mathrm{D}_k(A)\xrightarrow{\delta\otimes_A A/fA}\mathfrak{m}_A/f\cdot\mathfrak{m}_A\to\mathfrak{m}_A/fA$$

of canonical surjective A/fA-homomorphisms which we will (by abuse of notation) denote by $\bar{\delta}:\mathrm{D}_k(A)/f\mathrm{D}_k(A)\to\mathfrak{m}_A/fA$. Similarly we denote the canonical surjective map $\delta^{**}\otimes_A A/fA:(\mathrm{D}_k(A))^{**}/f(\mathrm{D}_k(A))^{**}\to A/fA$ by $\bar{\delta}^{**}$. For sake of clarity we denote by ′ the functor $\mathrm{Hom}_{A/fA}(-,A/fA)$. The map $\bar{\delta}^{**\prime\prime}:((\mathrm{D}_k(A))^{**}/f(\mathrm{D}_k(A))^{**})''\to A/fA$ is surjective, and by the special choice of f we obtain that the canonical (A/fA)-linear map

$$(\mathrm{D}_k(A)/f\mathrm{D}_k(A))''\to((\mathrm{D}_k(A))^{**}/f(\mathrm{D}_k(A))^{**})''$$

is bijective, since it is bijective in all prime ideals of height 2 in A which contain fA (observe $f\notin\bigcup\mathfrak{Q}$!). We obtain that the composite map

$$(\mathrm{D}_k(A)/f\mathrm{D}_k(A))''\xrightarrow{\sim}((\mathrm{D}_k(A))^{**}/f(\mathrm{D}_k(A))^{**})''\to A/fA$$

is surjective and that its restriction on $\mathrm{D}_k(A)/f\mathrm{D}_k(A)$ is the canonical map $\bar{\delta}:\mathrm{D}_k(A)/f\mathrm{D}_k(A)\to\mathfrak{m}_A/fA$. It follows that $\bar{\delta}$ generates a free direct summand of $(\mathrm{D}_k(A)/f\mathrm{D}_k(A))'$, see the proof of Lemma 2.1. Now, let

$$\varphi:\mathrm{D}_k(A)/f\mathrm{D}_k(A)\to\mathrm{D}_k(A/fA)$$

denote the canonical surjection and

$$\psi:(\mathrm{D}_k(A/fA))'\to(\mathrm{D}_k(A)/f\mathrm{D}_k(A))'$$

the canonical injection. If $\Delta:A/fA\to A/fA$ denotes the derivation which is induced by δ, then $\varphi'(\Delta) = \bar{\delta}$, since these maps agree on a suitable system of generators of $D_k(A)/fD_k(A)$:

$$\varphi'(\Delta)\overline{dx_i} = (\Delta \circ \varphi)\overline{dx_i} = m_i \cdot \bar{x}_i = \bar{\delta}(\overline{dx_i}).$$

Since $\bar{\delta}$ is a direct A/fA-summand of $(D_k(A)/fD_k(A))'$ it follows the same for Δ in $(D_k(A/fA))'$. This completes the proof of Proposition 2.4.

2.5. Remark. We do not know whether in Proposition 2.4 the condition 'Gorenstein' can be weakened to the condition 'Cohen–Macaulay'. We have proved Proposition 2.4 by showing the (perhaps) stronger assertion that the grading derivation δ does *not* generate a free direct summand of the derivation module $(D_k(A))^*$ of A, and hence we have a factorization (see 2.1 and 2.2):

$$D_k(A) \to (D_k(A))^{**} \xrightarrow{\delta^{**}} \mathfrak{m}_A.$$

The reduction to the two-dimensional case uses not the Gorenstein hypothesis itself, but only the hypothesis that A is a (normal) Macaulay-ring. So, if one could *disprove* a canonical decomposition $(D_k(A))^* = A \cdot \delta \oplus \omega_A^*$ in the two-dimensional normal *non-Gorenstein* case, too, the assertion of Proposition (2.4) would still be valid, if one replaces the hypothesis 'Gorenstein' by 'Cohen–Macaulay'. It seems likely that in the non-Gorenstein case, too, the grading derivation δ cannot span a free direct summand of $(D_k(A))^*$, although, however, by the Lemma of Zariski δ is always part of a minimal system of generators of $(D_k(A))^*$, A being a non-regular isolated singularity; in this case the Lemma of Zariski says that all derivations of A map $\mathfrak{m}_A$ into itself, see, for instance, the remark at the end of Proposition 2 in [6], and from $\delta \in \mathfrak{m}_A(D_k(A))^*$ we would obtain: $m_1 x_1 = \delta x_1 \in \mathfrak{m}_A^2$, a contradiction.

3. Invariant subrings

In this section let B be a convergent power series ring over the (valued) field k and G be a finite group of k-algebra-automorphisms on B with card G being a unit in k. By Chap. III, §3, Satz 2 in [2] there exists a regular system of parameters $X_1, \ldots, X_s$ of B such that G acts linearily in $X_1, \ldots, X_s$. Thus the invariant ring B^G has the form $A = k\langle F_1, \ldots, F_m \rangle$, F_i being homogeneous of degree c_i. If $c_i \neq 0$ in k, $i = 1, \ldots, m$, then A is a local graded analytic k-algebra. In [5, p. 4] we have shown that one can find always elements $F_1, \ldots, F_t \in \{F_1, \ldots, F_m\}$ such that $F_1, \ldots, F_t$ generate a $\mathfrak{m}_A$-primary ideal in A and $c_i \neq 0$ in k, $i = 1, \ldots, t$. From this fact we deduced that there exist $s = \dim A$ differentials $dF_1, \ldots, dF_s$ which form part of minimal system of generators of $(D_k(A))^{**}$, see also Remark 5 in [6]. In case char $k = 0$ the following proposition has been proved in [5, 2.7]; for sake of completeness we include this case here, too.

3.1. Proposition. *Let G be a finite group of k-algebra-automorphisms on the convergent power series ring B over k and $A := B^G$ be the invariant analytic k-algebra. Assume that one of the following conditions is satisfied:*

(i) *k has characteristic zero.*

(ii) *G is abelian and* card G *is a unit in k.*

Then the following assertions hold:

(1) *The canonical induced map $\mathrm{D}_k(A)\otimes_A k_A \to (\mathrm{D}_k(A))^{**}\otimes_A k_A$ is injective.*

(2) *The torsion-submodule of $\mathrm{D}_k(A)$ is contained in $\mathfrak{m}_A \mathrm{D}_k(A)$.*

(3) *$\mu(\mathrm{D}_k(A))^{**} = \mu(\mathrm{D}_k(B))^G = \mu(\mathrm{D}_k(A)) + \mu(\mathrm{D}_A(B))^G$, where $\mu(\ )$ denotes the minimal number of generators.*

Proof. Only assertion (1) has to be proved, since (2) is an easy consequence, and (3) will follow from (1) by [5, 2.3]. Let $X_1, \dots, X_s$ be a regular system of parameters of B on which G acts linearily. Since $\mathrm{D}_k(B)$ is a reflexive A-module, the canonical map $\mathrm{D}_k(A)\to\mathrm{D}_k(B)$ factors through $(\mathrm{D}_k(A))^{**}$:

$$\mathrm{D}_k(A)\to(\mathrm{D}_k(A))^{**}\to\mathrm{D}_k(B).$$

Therefore it only remains to be shown that the canonical induced map

$$(*)\qquad \mathrm{D}_k(A)\otimes_A k_A\to\mathrm{D}_k(B)\otimes_A k_A$$

is injective. Let $\delta:\mathrm{D}_k(B)\to\mathfrak{m}_B$ denote the B-homomorphism with $\delta(\mathrm{d}X_i)=X_i$. If $A = k\langle F_1,\dots,F_m\rangle$ with F_i being a minimal system of homogeneous generators of $\mathfrak{m}_A$ with degree c_i, then the composition

$$\mathrm{D}_k(A)\to\mathrm{D}_k(B)\xrightarrow{\delta}\mathfrak{m}_B\xrightarrow{\pi}\mathfrak{m}_A\quad\text{with } \pi:=(\operatorname{card} G)^{-1}\sum_{\tau\in G}\tau$$

maps $\mathrm{D}_k(A)$ *onto* $(c_1F_1,\dots,c_mF_m)A$. Thus in case char $k=0$ the assertion is true.

Now we will prove the abelian case. In order to prove that the induced map $(*)$ is injective, we may assume that $A=\hat{A}$ and $\hat{B}=B\otimes_A\hat{A}$ are complete. If $k'\supseteq k$ is an algebraically closed field and $A':=A\hat{\otimes}_k k'$, then G acts on $B':=B\otimes_A A'$ with invariant ring A'. If

$$\mathrm{D}_{k'}(A')\otimes_{A'}k_{A'}\to\mathrm{D}_{k'}(B')\otimes_{A'}k_{A'}$$

is injective with $\mathrm{D}_{k'}(A')=\mathrm{D}_k(A)\otimes_A A'$ and $\mathrm{D}_{k'}(B')=\mathrm{D}_k(B)\otimes_A A'$, cf. [7, 2.7], it follows the same for the map $(*)$. Therefore we may assume that $k=k'$ is algebraically closed. Then, since G is abelian with card G being a unit in k, the regular system of parameters $X_1,\dots,X_s$ may be chosen in such a way that *all* elements of G are diagonal operators in $X_1,\dots,X_s$. Thus the maximal ideal $\mathfrak{m}_A$ of A is minimally generated by monomials $M_1,\dots,M_m$. Let $p:=\text{char } k\geq 2$. First we show that the partial derivatives of M_1 (resp. M_i) cannot vanish simultaneously. Let us assume:

$$\partial_1 M_1=\partial_2 M_1=\cdots=\partial_s M_1=0.$$

Then there exists a monomial $M \in \mathfrak{m}_B$ with $M_1 = M^p$.
It follows for $\tau \in G$:

$$0 = \tau(M_1) - M_1 = \tau(M^p) - M^p = (\tau(M) - M)^p$$

and hence $\tau(M) = M$ for all $\tau \in G$, and therefore $M \in \mathfrak{m}_A$ and $M_1 = M^p \in \mathfrak{m}_A^2$. This is a contradiction. We obtain that there exist elements $c_{ij} \in k$ with

$$c_{ij} M_j = \partial_i M_j \cdot X_i, \qquad i = 1, \ldots, s, \quad j = 1, \ldots, m,$$

where for any fixed index $j \in \{1, \ldots, m\}$ there exists an index $i \in \{1, \ldots, s\}$ with $c_{ij} \neq 0$.

Now assume that there exist elements $a_j \in A$ with:

$$\mathrm{d}M_1 - \sum_{j=2}^{m} a_j \mathrm{d}M_j \in \mathfrak{m}_A \mathrm{D}_k(B)$$

which implies

$$\partial_i M_1 - \sum_{j=2}^{m} a_j \partial_i M_j \in \mathfrak{m}_A B, \quad i = 1, \ldots, s.$$

Let $c_{11} \neq 0$, then it follows

$$\partial_1 M_1 \cdot X_1 \cdot c_{11}^{-1} - \sum_{j=2}^{m} a_j c_{11}^{-1} \partial_1 M_j \cdot X_1 \in \mathfrak{m}_A \cdot \mathfrak{m}_B$$

or

$$M_1 - \sum_{j=2}^{m} a_j c_{11}^{-1} \cdot c_{1j} M_j \in \mathfrak{m}_A \cdot \mathfrak{m}_B.$$

Now, if we apply the Reynolds-operator $(\operatorname{card} G)^{-1} \sum_{\tau \in G} \tau$ to this relation, we get:

$$M_1 - \sum_{j=2}^{m} a_j c_{11}^{-1} c_{1j} M_j \in \mathfrak{m}_A^2,$$

which contradicts the fact that the monomials $M_1, \ldots, M_m$ are a minimal system of generators of $\mathfrak{m}_A$. Thus the abelian case has been proved, too.

3.2. Remark. Let G, $B = k\langle X_1, \ldots, X_s \rangle$ and $A = k\langle F_1, \ldots, F_m \rangle$ be as in the beginning of this section and $\delta : \mathrm{D}_k(B) \to \mathfrak{m}_B$ be the B-linear map with $\delta(\mathrm{d}X_i) = X_i$. Since G acts linearily in $X_1, \ldots, X_s$, one easily checks that δ is a G-homomorphism, and therefore

$$\delta^G : (\mathrm{D}_k(B))^G \to (\mathfrak{m}_B)^G = \mathfrak{m}_A$$

is surjective, too. Now, it has been proved in [5, 2.3] that the canonical map

$$(\mathrm{D}_k(A))^{**} \to ((\mathrm{D}_k(B))^G)^{**} = (\mathrm{D}_k(B))^G$$

is bijective. Hence we have a surjective map

$$(\mathrm{D}_k(A))^{**} \to \mathfrak{m}_A$$

whose restriction on $D_k(A)$ acts on dF_i as $(\deg F_i)\cdot F_i$. In case $\deg F_i \neq 0$ in k, $i=1,\dots,m$, we obtain from Remark 2.2 that $\delta\,|\,D_k(A)$ is *not* a direct summand of $(D_k(A))^*$, even in case of a non-Gorenstein invariant ring $A=B^G$. *This shows that any further investigation of problems raised in Remark 2.5 requires a study of two-dimensional normal graded non-Gorenstein algebras which are not invariant rings.*

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